

# Reference spaces in Special Relativity Theory: an intrinsic approach

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## Abstract

Starting from a suggestion of Einstein on the construction of the concept of space, we elaborate an intrinsic method to obtain space and time transformations between two inertial spaces of reference, mathematically modeled as affine euclidean spaces. The principal device introduced for relating the space readings in both spaces is the so-called *tracer mapping*, which makes a snapshot of a space onto the other. The general form of the space and time transformations is obtained as an affine-preserving mapping compatible with the principle of relativity, a cylindrical symmetry around the relative velocities between spaces and the group character of the transformations. After having obtained Galileo and Lorentz transformations, the same method has been applied to two classical problems: the Coriolis theorem of Newtonian Mechanics and the geometry of a rotating disk in Special Relativity. Even in the case of Newtonian Mechanics, the possibility of distinguishing the spaces of reference is found useful.

KEY WORDS : Concept of Space, Space and Time Transformations, Reference Frames.

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# 1 Introduction

The nature of motion is an important issue since the first stages in the science of mechanics. As it happened with many other topics, Galileo was one of the first to deal with this problem in an inquiring form. In his celebrated treatise *Dialogue Concerning the Two Chief World Systems Ptolemaic and Copernican*, Galilei (1632), he faced the problem of the relativity of the motion and argued that one may refer a motion not only to the resting Earth but also to other bodies moving with respect to it. Gradually, it was becoming clear that, for studying a motion, use had to be made only of the space relative to a body of reference, making thus needless even to mention an absolute space. In spite of this knowledge being so old, its operational meaning was not fully understood until Einstein's findings on General Relativity. In his book *The Meaning of Relativity*, Einstein (1953), he set out clearly his ideas on the relativity of the space:

“For the concept of space the following seems essential. We can form new bodies by bringing bodies  $B$ ,  $C$ , ... up to body  $A$ ; we say that we *continue* body  $A$ . We can continue body  $A$  in such a way that it comes into contact with any other body,  $X$ . The ensemble of all continuations of body  $A$  we can designate as the ‘space of the body  $A$ .’ Then it is true that all bodies are in the ‘space of the (arbitrarily chosen) body  $A$ .’ In this sense we cannot speak of space in the abstract, but only of the ‘space belonging to a body  $A$ .’ The earth’s crust plays such a dominant role in our daily life in judging the relative positions of bodies that it has led to an abstract conception of space which certainly cannot be defended. In order to free ourselves from this fatal error we shall speak only of ‘bodies of reference,’ or ‘space of reference.’ ”

According to Einstein, the space of a body is a physical concept and, as such, operationally defined. He emphasized several features of it:

1. Any body may be used as a reference body. In special relativity we will limit ourselves to inertial bodies.
2. The points of a reference space have a permanent character to the extent that they can be conceived as particles of some continuation of the body. The points of a space of reference are by definition at rest.

3. In a given space of reference, the record of the position of a moving particle is the point of that body which coincides with it at the considered time instant. There are no moving *points* but moving *particles* passing by close points in a space of reference, instead.

The fact that the space is relative in no way means that this concept is irrelevant in physics. Energy or electric field are also relative concepts but however we do not dismiss them out as of lacking interest. Even though this is clear enough, after the Minkowskian geometric interpretation of spacetime the use of particular bodies of reference has been scorned as an old pre-relativistic prejudice similar to absolute time or instantaneous action at a distance. Nevertheless, to experimentally support a physical theory, it has usually to recourse to data obtained in a certain laboratory space. Therefore, it seems to be of the utmost interest to explicit the usage rules for spaces of reference in the realm of special relativity.

Besides the customary method based upon the use of coordinates, the study of spacetime can be undertaken in an axiomatic mathematically rigorous way in which it is considered as an affine pseudoeuclidean space. Matolcsi (1993) has promoted such a procedure in order to build up an intrinsic formalism upon which any physical theory can be erected. Our viewpoint is complementary: we are concerned with the construction of spacetime from the analysis of the transformations between inertial reference spaces.

Thus, the objectives this work is aimed to are:

1. To cast in mathematical form the idea of space of reference devised by Einstein.
2. Since the geometrical and physical relations on each body of reference do not depend on the chosen coordinate system, we will develop an intrinsic method, directly based on absolute objects of the Euclidean affine tridimensional space. The relativity of space demands to consider a different affine space associated to each body.
3. To present a theoretical device, that we call tracer mapping, having a direct physical interpretation, suited to characterize the relationship between the physical quantities measured in different spaces, without coordinates. The concept of tracer is not new, but we think it has not received the attention it deserves. For instance, Synge (1972) introduced the term *snapshot*, taken from Milne's *world-map*, a term used in the same sense by Rindler (1977).

4. To obtain *ab initio* intrinsic expressions of the Galileo and Lorentz transformations.

The proposed method allows to wholly solve, or avoid, some little problems appearing in the usual derivations of space and time transformations, which are really associated to a particular choice of a coordinate system in each reference body. For instance we could list: the problem of parallelism between coordinate axes, addressed in Frahm (1979); the reciprocity relation for the relative motion of two inertial frames of reference, analysed in detail in Berzi *et al.* (1968); and the search for Lorentz transformations in cases with arbitrarily oriented axes, carefully studied in Cushing (1967).

In order to state clearly the components used in the construction of the space and time transformations, we have resorted to a style akin to the “more geometrico”. However, this work is not addressed to a search for a minimum set of postulates necessary for a logically closed derivation [for that purpose see, for example, Matolcsi (1993), Lévy–Leblond (1976), Nishikawa (1997), Schwartz (1984) and References cited therein]. Instead, the intention of this work is of a methodological nature. We try to clarify as far as possible the ingredients of physical character which arise on occasion of the analysis of the problems concerning the description of the measurements in space and time carried out by several observers and the comparison between these measurements.

This paper is organized as follows. In Section 2, using the tracer mapping concept, the intrinsic method is introduced. By adding further physical hypotheses, we obtain in Sections 3 and 4 the transformations of Galileo and Lorentz. Next Section is devoted to widen the scope of the intrinsic method by giving an coordinate-free definition of angular velocity leading to Coriolis theorem, within the limits of Newtonian mechanics, and a characterization of the geometry on a rotating disk, in special relativity.

## 2 The intrinsic method

### SPACES, TIMES AND EVENTS

According to Einstein’s construction, to every body  $A$  there corresponds a space of reference  $K$ . Besides, if the body is an inertial one, its space is affine and euclidean. At every point, the spaces are equipped with identical measuring rods, i. e., rods made by using the same instructions, which define the same length unit in all of them.

From now on, the word “space” as used in this paper will mean the space of reference of a body. In that sense, a space has an objective character and its points are permanent as they can be considered as particles of a certain body.

Unless otherwise stated, only inertial bodies and spaces of reference will be considered. For any of these spaces of reference, such as  $K$ , being an affine space, the set of its translations (pairs of points) forms a vector space called  $\mathcal{V}$ . Following definitions and notation given in Crampin *et al.* (1986), Chap. 1, we will represent a translation of the point  $Q$  by the vector  $\mathbf{v}$  as a new point  $P$  given by the sum  $Q + \mathbf{v}$ .

On each space of reference identical clocks, which define the same time measuring unit in them, are distributed. If the space is inertial, these clocks can be synchronized once and for all.

**Postulate 1** *Each event  $\mathcal{P}$  happens in the neighborhood of a single point  $P$  of the space  $K$ . The clock at  $P$  registers its date,  $t$ , a single real number. The record  $(t, P)$  fully characterizes the event.*

In a similar way, the same event  $\mathcal{P}$  is recorded in the space  $K'$  of other body as  $(t', P')$ . Two different records in  $K$  represent two different events that will produce also different records in any other space. This circumstance confers the event manifold an absolute character.

It is to be emphasized that  $(t, P)$  is a very primitive way to register an event. This record consists of a point and a real number rather than four coordinates.

#### PARTICLES, TRAJECTORIES AND VELOCITIES

A particle  $M$  produces in the space  $K$  the set of records  $(t, P_t)$ ,  $-\infty < t < +\infty$ , called the trajectory of  $M$ , a curve in  $K$ .

**Definition 1** *The velocity  $\mathbf{v}$  of  $M$  with respect to  $K$  at time  $t$  is*

$$\mathbf{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{P_t P_{t+\Delta t}}}{\Delta t}. \quad (1)$$

Hence,  $\mathbf{v}$  is a vector belonging to the space  $\mathcal{V}$  associated to  $K$  and, thus, it is a relative quantity obtained from a series of permanent records in  $K$ .

#### SPACE AND TIME TRANSFORMATIONS

The study of the space and time transformations following Einstein's guidelines can be undertaken by using the theoretical devices introduced below.

**Definition 2** *The bijective mapping  $\Lambda_t : K' \longrightarrow K$  maps each point  $P' \in K'$  onto the point  $P \in K$  which mets  $P'$  in the instant  $t$ . We will refer  $\Lambda_t$  to as tracer mapping.*

The suitability of this definition of  $\Lambda_t$  as a bijective mapping will be evident upon its explicit construction. We call *trace* at  $t$  of the point set  $\Omega' \subset K'$  on the space  $K$  to the set  $\Omega = \Lambda_t(\Omega')$ . The trace, being a set of points in  $K$ , has a permanent character and it could be envisaged as a contact photograph of the space  $K'$ . Also, note that, in order to get the trace, no time consideration in  $K'$  has to be made.

**Definition 3** *The event  $\mathcal{P}$ , characterized at  $K$  by the pair  $(t, P)$ , happens in  $K'$  at the date  $t'$ . Therefore, we can introduce a function  $f'$  defined as*

$$t' = f'(t, P). \quad (2)$$

*The function  $f'_P : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f'_P(t) = f'(t, P)$  is invertible.*

The pair of relations

$$\begin{cases} P &= \Lambda_t(P'), \\ t' &= f'(t, P), \end{cases} \quad (3)$$

will be called mixed space and time transformation formulas (note the prime on those mappings with image on  $K'$ ). They suffice for relating the records of any event in the spaces  $K$  and  $K'$ . The mixed formulas are equivalent to the standard ones, in which  $(t', P')$  is obtained from  $(t, P)$ . In fact, since  $\Lambda_t$  has an inverse, call it  $(\Lambda_t)^{-1}$ , the first of the Equations (3) leads to  $P' = (\Lambda_t)^{-1}(P)$  which, together with  $t' = f'(t, P)$ , constitutes the formulas for the standard transformation. As we will see, the form (3) will prove to be very suitable for our intrinsic derivation. Moreover, it will be seen how certain symmetry considerations are all we need to carry them to their final form.

From the invertibility of the mappings  $\Lambda_t$  and  $f'_P$ , stated in their respective definitions, it follows that the mixed transformation formulas as a whole are also invertible. In fact, if  $(\Lambda_t)^{-1}$  and  $(f'_P)^{-1}$  do exist, then

$$\begin{cases} P' &= (\Lambda_t)^{-1}(P), \\ t &= (f'_P)^{-1}(t'), \end{cases} \quad (4)$$

allowing to write that

$$P' = \left( \Lambda_{(f'_P)^{-1}(t')} \right)^{-1} (P), \quad (5)$$

which can be identified with the expression representing the trace of  $K$  on  $K'$  at the time  $t'$ ,  $P' = \Lambda'_{t'}(P)$ , i. e., with the first of the equations of the inverse mixed transformation. Resorting in advance to the principle of relativity, the spaces  $K$  and  $K'$  are on equal terms and, therefore, in the same way as  $\Lambda_t$ , the tracer mapping  $\Lambda'_{t'}$  must also be invertible, so that

$$P = (\Lambda'_{t'})^{-1} (P'). \quad (6)$$

By substitution of this expression for  $P$  in the second of the equations (4), we obtain

$$t = \left( f'_{(\Lambda'_{t'})^{-1}(P')} \right)^{-1} (t'), \quad (7)$$

which can be identified with the second of the equations for the inverse mixed transformation,  $t = f_{P'}(t')$ .

Since the standard transformation formulas can be derived from the mixed, the invertibility of the latter implies that of the former, as it was to be expected.

#### SPATIAL HOMOGENEITY OF THE TRANSFORMATION

As previously stated, the inertial character of  $K$  and  $K'$  implies that both spaces are homogeneous and isotropic. On the other hand, the transformation among spaces and times itself may not depend on the considered point. This property, which we will call spatial homogeneity of the transformation, is expressed as it follows:

**Postulate 2** *The mappings  $\Lambda_t$  and  $f'$  are homogeneous in their spaces, i. e., they do not privilege any of their points.*

Consider first the mapping  $\Lambda_t$ . Since  $K'$  is affine, we have the obvious identity

$$\Lambda_t(Q' + \mathbf{r}') = \Lambda_t(Q') + \overrightarrow{\Lambda_t(Q')\Lambda_t(Q' + \mathbf{r}')}, \quad (8)$$

where the vector  $\overrightarrow{\Lambda_t(Q')\Lambda_t(Q' + \mathbf{r}')}$ , being a function of  $t$ ,  $Q'$  and  $\mathbf{r}'$ , will be written as  $\lambda_t(Q', \mathbf{r}')$ . But Postulate 2 means that a vector  $\mathbf{r}' \in \mathcal{V}'$  placed

at two different points of  $K'$  must map into the same vector in  $\mathcal{V}$ . Thus,  $\lambda_t(Q', \mathbf{r}') = \lambda_t(R', \mathbf{r}')$ , for any pair of points  $Q'$  and  $R'$ , and so the argument  $Q'$  in  $\lambda_t$  can be suppressed. Instead of Eq. (8) we have, then,

$$\Lambda_t(Q' + \mathbf{r}') = \Lambda_t(Q') + \lambda_t(\mathbf{r}'). \quad (9)$$

Now, we can show that  $\lambda_t$  is a linear mapping because of  $K'$  being an affine space. In fact, consider two successive translations in  $K'$ ,  $\mathbf{r}'$  and  $\mathbf{s}'$ , for which  $(Q' + \mathbf{r}') + \mathbf{s}' = Q' + (\mathbf{r}' + \mathbf{s}')$ , and

$$\begin{aligned} \Lambda_t[Q' + (\mathbf{r}' + \mathbf{s}')] &= \Lambda_t(Q') + \lambda_t(\mathbf{r}' + \mathbf{s}') \\ &= \Lambda_t[(Q' + \mathbf{r}') + \mathbf{s}'] = \Lambda_t(Q' + \mathbf{r}') + \lambda_t(\mathbf{s}') \\ &= \Lambda_t(Q') + \lambda_t(\mathbf{r}') + \lambda_t(\mathbf{s}'). \end{aligned}$$

Thus,  $\lambda_t(\mathbf{r}' + \mathbf{s}') = \lambda_t(\mathbf{r}') + \lambda_t(\mathbf{s}')$ . Starting with identities like

$$\lambda_t(2\mathbf{r}') = \lambda_t(\mathbf{r}' + \mathbf{r}') = \lambda_t(\mathbf{r}') + \lambda_t(\mathbf{r}') = 2\lambda_t(\mathbf{r}'),$$

it is easily shown, for an  $m$  being successively integer, rational and real, that  $\lambda_t(m\mathbf{r}') = m\lambda_t(\mathbf{r}')$ , thus finishing the proof.

With respect to  $f'$ , we can follow a similar line of reasoning. In the obvious identity

$$f'(t, Q + \mathbf{r}) = f'(t, Q) + [f'(t, Q + \mathbf{r}) - f'(t, Q)] = f'(t, Q) + \theta'_t(Q, \mathbf{r}),$$

the assumed spatial homogeneity in  $f'$  implies that  $\theta'_t(Q, \mathbf{r})$  must be independent on the particular point  $Q$ , then

$$f'(t, Q + \mathbf{r}) = f'(t, Q) + \theta'_t(\mathbf{r}). \quad (10)$$

By combining two translations in  $K$  we can show that  $\theta'_t(\mathbf{r})$  is linear. In fact,

$$\begin{aligned} f'[t, Q + (\mathbf{r} + \mathbf{s})] &= f'(t, Q) + \theta'_t(\mathbf{r} + \mathbf{s}) \\ f'[t, (Q + \mathbf{r}) + \mathbf{s}] &= f'(t, Q + \mathbf{r}) + \theta'_t(\mathbf{s}) = f'(t, Q) + \theta'_t(\mathbf{r}) + \theta'_t(\mathbf{s}), \end{aligned}$$

hence  $\theta'_t(\mathbf{r} + \mathbf{s}) = \theta'_t(\mathbf{r}) + \theta'_t(\mathbf{s})$ . And, finally, as in the case of  $\lambda_t$ ,  $\theta'_t(m\mathbf{r}) = m\theta'_t(\mathbf{r})$ .

The same reasoning can now be used with the other argument in  $f'$ . In the identity

$$f'(t + \tau, Q) = f'(t, Q) + [f'(t + \tau, Q) - f'(t, Q)],$$



the term between square brackets, in principle a function of  $t, \tau$  and  $Q$ , actually cannot depend on  $Q$  according to the postulated homogeneity of  $f'$ , therefore

$$f'(t + \tau, Q) = f'(t, Q) + g'(t, \tau),$$

where  $g'(t, \tau)$  depends on  $t$  and linearly on  $\tau$ , i. e.,  $g'(t, \tau) = \gamma'(t)\tau$ . This last statement can be shown by considering the composition of two arbitrary translations in time. We get the relation

$$f'(t + \tau, Q + \mathbf{r}) = f'(t, Q) + \gamma'(t)\tau + \theta'_t(\mathbf{r}). \quad (11)$$

### TIME HOMOGENEITY OF THE TRANSFORMATION

In a similar way as in the last Subsection, the transformation between spaces of reference may not depend on a particular chosen time. This property, that we will call time homogeneity of the transformation, is expressed as it follows:

**Postulate 3** *The mappings  $\Lambda_t$  and  $f'$  are homogeneous in time, i. e., they do not privilege any time instant.*

An arbitrary point  $P'$  of  $K'$  describes in  $K$  the trajectory  $\Lambda_t(P')$ . The homogeneity in time of the mapping  $\Lambda_t$  implies that such a motion is uniform, i.e.,

$$\Lambda_t(P') = \Lambda_0(P') + \mathbf{u}(P')t. \quad (12)$$

Here,  $\mathbf{u}(P')$  is the velocity field linked to the particles in  $K'$ , the *material velocity*, in the language of the Kinematics of Continuous Media, see Marsden *et al.* (1994), p. 26. We may express this field also in terms of the points of  $K$  (usually called *spatial velocity*, see Marsden *et al.* (1994), p. 27):

$$\mathbf{U}(t, P) = \mathbf{u}[(\Lambda_t)^{-1}(P)]. \quad (13)$$

Since time is homogeneous for the transformation,  $K$  must observe a steady situation in any point and, thus, the space velocity field must be time independent:

$$\frac{\partial \mathbf{U}}{\partial t} = 0. \quad (14)$$

This condition can only be satisfied if  $\mathbf{u}(P')$  is the same for every point  $P'$ . It is worth noting that the relative motion between  $K$  and  $K'$  establishes a relationship between the homogeneities in time and in space  $K'$ .

By differentiating (9) with respect to time we obtain the relation

$$\mathbf{u}(Q' + \mathbf{r}') = \mathbf{u}(Q') + \frac{d\lambda_t}{dt}(\mathbf{r}'). \quad (15)$$

If, as seen, every point of  $K'$ , in particular  $Q'$  and  $Q' + \mathbf{r}'$ , have the same velocity, the following relation will hold

$$\frac{d\lambda_t}{dt} = 0, \quad (16)$$

that is,  $\lambda_t$  must be time independent and, thus, it can be represented simply as  $\lambda$ .

Two simultaneous events at  $Q$  and  $Q + \mathbf{r}$  happen in  $K'$  with a time separation that, according to Eq. (10), is given by

$$f'(t, Q + \mathbf{r}) - f'(t, Q) = \theta'_t(\mathbf{r}). \quad (17)$$

Time homogeneity in  $f'$  requires that the above separation does not depend on the chosen instant,  $t$ . Therefore, the index in  $\theta'_t$  is not necessary, so that

$$f'(t, Q + \mathbf{r}) = f'(t, Q) + \theta'(\mathbf{r}). \quad (18)$$

On the other hand, for two events happening in the same point,  $Q$ , with a time separation,  $\tau$ , we have, from Eq. (11),

$$f'(t + \tau, Q) - f'(t, Q) = \gamma'(t)\tau. \quad (19)$$

In the same way, the condition for time homogeneity leads to  $\gamma'$  to be independent of  $t$ . The results concerning the last two Postulates can be summarized in the following theorem:

**Theorem 1** *The homogeneities in space and time of the mappings  $\Lambda_t$  and  $f'$  imply*

$$\Lambda_t(Q' + \mathbf{r}') = \Lambda_0(Q') + \mathbf{u}t + \lambda(\mathbf{r}') \quad (20)$$

$$f'(t + \tau, Q + \mathbf{r}) = f'(t, Q) + \gamma'\tau + \theta'(\mathbf{r}), \quad (21)$$

where  $\mathbf{u}$  is the velocity of any particle in  $K'$  as measured in  $K$ ,  $\lambda$  is a linear mapping of  $\mathcal{V}'$  on  $\mathcal{V}$ ,  $\gamma'$  is a number and  $\theta'$  is a covector in  $\mathcal{V}^*$ .

In order to take advantage of the affine property of the spaces we find useful to use vectors to represent points, which is carried out by choosing a point and an instant as an origin in each space and time. This process can be independently made in each space but the theory becomes simpler if one chooses a single, though arbitrary, event  $\mathcal{Q}$  as a reference or origin. This event registers at  $K$  and  $K'$  as  $(t_{\mathcal{Q}}, Q)$  and  $(t'_{\mathcal{Q}}, Q')$ , respectively, and we have the relations

$$\begin{cases} Q &= \Lambda_{t_{\mathcal{Q}}}(Q'), \\ t'_{\mathcal{Q}} &= f'(t_{\mathcal{Q}}, Q). \end{cases} \quad (22)$$

The choice of a reference event is the first step to coordinate the space and time records in both spaces of reference.

An arbitrary event,  $\mathcal{P}$ , is registered as  $(t_{\mathcal{Q}} + \tau, Q + \mathbf{r})$  and  $(t'_{\mathcal{Q}} + \tau', Q' + \mathbf{r}')$  at  $K$  and  $K'$ , respectively. Hence, we can write

$$\begin{cases} Q + \mathbf{r} &= \Lambda_{t_{\mathcal{Q}}}(Q' + \mathbf{r}'), \\ t'_{\mathcal{Q}} + \tau' &= f'(t_{\mathcal{Q}} + \tau, Q + \mathbf{r}). \end{cases} \quad (23)$$

Then, by using Eqs. (20) and (21), the mixed transformation formulas (3) can be written in terms of vector quantities as

$$\mathbf{r} = \mathbf{u}\tau + \lambda(\mathbf{r}') \quad (24)$$

$$\tau' = \gamma'\tau + \theta'(\mathbf{r}). \quad (25)$$

#### SOME CONSEQUENCES OF THE PRINCIPLE OF RELATIVITY

The inverse transformation formulas,

$$\begin{cases} P' &= \Lambda'_{t'}(P), \\ t &= f(t', P'), \end{cases} \quad (26)$$

are merely those obtained by exchanging  $K$  and  $K'$ . According to the principle of relativity, both spaces of reference are physically equivalent. Therefore we are allowed to repeat the previous analysis but now from the point of view of  $K'$ . Thus, we arrive to the vector relations, similar to (24) and (25),

$$\mathbf{r}' = \mathbf{u}'\tau' + \lambda'(\mathbf{r}) \quad (27)$$

$$\tau = \gamma\tau' + \theta(\mathbf{r}'). \quad (28)$$

Here,  $\mathbf{u}' \in \mathcal{V}'$  represents the velocity of the particles of  $K$  as seen from  $K'$ .

**Theorem 2** *The mappings and constants  $\lambda$ ,  $\lambda'$ ,  $\theta$ ,  $\theta'$ ,  $\gamma$  and  $\gamma'$  satisfy the identities*

$$\mathbf{u} + \gamma' \lambda(\mathbf{u}') = 0, \quad (29)$$

$$\mathbf{r} = \theta'(\mathbf{r}) \lambda(\mathbf{u}') + \lambda[\lambda'(\mathbf{r})], \quad \forall \mathbf{r} \in \mathcal{V}, \quad (30)$$

$$\gamma'[\gamma + \theta(\mathbf{u}')] = 1, \quad (31)$$

$$\theta'(\mathbf{r}) + \gamma' \theta[\lambda'(\mathbf{r})] = 0, \quad \forall \mathbf{r} \in \mathcal{V}, \quad (32)$$

*and those resulting by interchanging the positions of primed and unprimed symbols*

$$\mathbf{u}' + \gamma \lambda'(\mathbf{u}) = 0, \quad (33)$$

$$\mathbf{r}' = \theta(\mathbf{r}') \lambda'(\mathbf{u}) + \lambda'[\lambda(\mathbf{r}')], \quad \forall \mathbf{r}' \in \mathcal{V}', \quad (34)$$

$$\gamma[\gamma' + \theta'(\mathbf{u}')] = 1, \quad (35)$$

$$\theta(\mathbf{r}') + \gamma \theta'[\lambda(\mathbf{r}')] = 0, \quad \forall \mathbf{r}' \in \mathcal{V}'. \quad (36)$$

These results are obtained by imposing the condition that (27) and (28) are the inverse ones of (24) and (25), and then by substitution of the former into the later ones.

**Postulate 4** *The following equalities hold*

$$\gamma' = \gamma, \quad (37)$$

$$u' = u, \quad (38)$$

where  $u' = |\mathbf{u}'|$  and  $u = |\mathbf{u}|$ .

This statement presupposes that identical length and time measuring standards have been adopted in both spaces, and it derives, then, from the principle of relativity. We can make this connection plausible by means of conceptual experiments like the following ones. In relation to Eq. (37), consider the measurement of the period of a standard clock, like a neutron lifetime, at rest in the origin of  $K$ . Let  $\Delta\tau_0$  and  $\Delta\tau'$  be the values obtained at  $K$  and  $K'$ , respectively. Then, according to Eq. (25),  $\Delta\tau' = \gamma' \Delta\tau_0$ . For the standard clock now at rest in  $K'$  one would obtain similarly  $\Delta\tau = \gamma \Delta\tau'_0$ , where  $\Delta\tau'_0$  and  $\Delta\tau$  are the measured period in  $K'$  and  $K$ , respectively. The equivalence between  $K$  and  $K'$ , implicit in the principle of relativity, implies (37); if not, a strange asymmetry between  $K$  and  $K'$  would appear possibly reminiscent

of an ether at rest, and both spaces could be distinguished from one another (see Nishikawa (1997) for a critical analysis of these assumptions and their relationship to the principle of relativity). Besides, if the times in both spaces flow in the same direction, all the time intervals considered so far are positive, and  $\gamma > 0$ . A similar argument involving the motion of the origins of both spaces can be used to justify Eq. (38).

#### CYLINDRICAL SYMMETRY OF THE TRANSFORMATION PROBLEM

**Postulate 5** *In the transformation problem, at  $K$ , the direction  $U$  of  $\mathbf{u}$  is privileged. All directions in the plane (subspace)  $T$  of vectors perpendicular to  $\mathbf{u}$  are physically equivalent. In the same way, at  $K'$ , the direction  $U'$  of  $\mathbf{u}'$  is privileged and all directions in the plane  $T'$  of vectors perpendicular to  $\mathbf{u}'$  are also physically equivalent.*

This cylindrical symmetry implies that, if  $\mathbf{r}_T \in T$ , then  $\theta'(\mathbf{r}_T)$  must be independent on the vector direction even though it will depend on its magnitude. To  $\lambda$ , this symmetry demands that, if  $\mathbf{r}'_T \in T'$ , then  $|\lambda(\mathbf{r}'_T)|^2$  is independent on the direction of  $\mathbf{r}'_T$ . Stated otherwise, the image by  $\lambda$  of a circumference of  $T'$  has to be a circumference in  $T$ .

**Theorem 3** *The action of the covector  $\theta'$  on arbitrary vectors is given by*

$$\theta'(\mathbf{r}) = \frac{1 - \gamma^2}{\gamma} \frac{\mathbf{r} \cdot \mathbf{u}}{u^2}. \quad (39)$$

Proof to Equation (39): Given any vector  $\mathbf{p}_1$  in the plane  $T$  we can add two more vectors in it,  $\mathbf{p}_2$  and  $\mathbf{p}_3$ , to form an equilateral triangle and such that

$$\mathbf{p}_1 = \mathbf{p}_2 - \mathbf{p}_3. \quad (40)$$

Since the magnitude of the three vectors is the same, the action of  $\theta'$  on any of them must be identical, i. e.,

$$\theta'(\mathbf{p}_1) = \theta'(\mathbf{p}_2) = \theta'(\mathbf{p}_3) = C.$$

Applying  $\theta'$  on both sides of Equation (40) and on account of its linearity, we conclude that  $C = 0$ , that is, the action of  $\theta'$  on any vector perpendicular to  $\mathbf{u}$  is zero. This result allows us to fully characterize  $\theta'$ . In fact, let  $\mathbf{r}_T$  and  $\mathbf{r}_U$  be the projections of a vector  $\mathbf{r}$ ,  $\mathbf{r} \in \mathcal{V}$ , on the plane  $T$  and on the direction of  $\mathbf{u}$ , respectively. As  $\mathbf{r}_U = (\mathbf{r} \cdot \mathbf{u}/u)(\mathbf{u}/u)$  and  $\theta'(\mathbf{r}_T) = 0$ , then  $\theta'(\mathbf{r}) = \theta'(\mathbf{r}_U)$ , and by using Equations (35) and (37) we obtain immediately Equation (39).

**Definition 4** We define the linear mappings  $\lambda_U : \mathcal{V}' \longrightarrow \mathcal{V}$  and  $\lambda_T : \mathcal{V}' \longrightarrow \mathcal{V}$  as

$$\begin{cases} \lambda_U(\mathbf{r}'_T) &= 0 \\ \lambda_U(\mathbf{r}'_U) &= \lambda(\mathbf{r}'_U) \end{cases} ; \quad \begin{cases} \lambda_T(\mathbf{r}'_T) &= \lambda(\mathbf{r}'_T) \\ \lambda_T(\mathbf{r}'_U) &= 0, \end{cases}$$

where  $\mathbf{r}'_T \in T'$  and  $\mathbf{r}'_U \in U'$ .

These definitions are consistent since  $\mathcal{V}' = U' \oplus T'$ , and it is straightforward to see that  $\lambda = \lambda_T + \lambda_U$ . In the same way, by introducing similar definitions for  $\lambda'$ , merely by exchanging primed and unprimed letters, we write  $\lambda' = \lambda'_T + \lambda'_U$ .

**Theorem 4** The linear mapping  $\lambda = \lambda_T + \lambda_U$  satisfies the conditions

$$\lambda_U(\mathbf{r}') = -\frac{1}{\gamma} \frac{\mathbf{r}' \cdot \mathbf{u}'}{u^2} \mathbf{u} \quad (41)$$

$$\mathbf{r}'_T \in T' \Rightarrow \lambda(\mathbf{r}'_T) \in T \quad (42)$$

$$\lambda_T(\mathbf{r}'_{T1}) \cdot \lambda_T(\mathbf{r}'_{T2}) = \mathbf{r}'_{T1} \cdot \mathbf{r}'_{T2}; \quad \mathbf{r}'_{T1}, \mathbf{r}'_{T2} \in T' \quad (43)$$

Equation (41) follows directly from Equation (29) and the result (37). The other ones can be shown from the cylindrical symmetry of the transformation and the principle of relativity, following a similar reasoning to the used to prove Theorem 3.

Equation (43) shows that  $\lambda_T$  preserves the magnitude of those vectors perpendicular to  $\mathbf{u}'$  when transformed from  $T'$  to  $T$ . As it is well known, a single parameter, usually an angle, is enough to determine this transformation. The value of this angle is so far undefined since we have not specified the orientation of  $K'$  with respect to  $K$ . Once the origins in spaces and times and the directions of the velocities  $\mathbf{u}$  and  $\mathbf{u}'$  have been established, the space  $K'$  can be rotated around  $\mathbf{u}'$  without any change in the previous results. From this consideration, it follows that no better determination of  $\lambda$  can be obtained unless that missing piece of information is added.

Finally, in order to fully characterize the mappings  $\theta'$  and  $\lambda_U$  we have to find out a relation between  $\gamma$  and  $\mathbf{u}$ . To that aim, we will extend the method developed for one-dimensional spaces in Lévy–Leblond (1976) and accordingly we will assume that the space and time transformations have a group structure. That group structure was already explored in part when we assumed the existence of an inverse transformation.

## CLOSURE OF COMPOSITION OF TRANSFORMATIONS

Let  $K_1$ ,  $K_2$  and  $K_3$  be three inertial spaces of reference. Between each pair of them a space and time transformation can be established:  $T^a$  between  $K_1$  and  $K_2$ ,  $T^b$  between  $K_2$  and  $K_3$ , and  $T^c$  between  $K_3$  and  $K_1$ . We assume, as before, that a single event defines origins in the three spaces and times. Each transformation is characterized by a relative velocity that can be measured in any of the two spaces related by the transformation. Thus,  $T^a$  is characterized by  $\mathbf{u}_1^a$  and  $\mathbf{u}_2^a$  in  $K_1$  and  $K_2$ , respectively. Let  $(\tau_1, \mathbf{r}_1)$  and  $(\tau_2, \mathbf{r}_2)$  be records of the same event at  $K_1$  and  $K_2$ , respectively. Our previous analysis [refer to Eqs. (24), (25), (37), (39) and (41), and to Definition 4] allows us to write the equations of the mixed transformation between those spaces as

$$\mathbf{r}_1 = \lambda_{T1}^a(\mathbf{r}_2) + \mathbf{u}_1^a \tau_1 - \frac{1}{\gamma_a} \frac{\mathbf{r}_2 \cdot \mathbf{u}_2^a}{u_a^2} \mathbf{u}_1^a, \quad (44)$$

$$\tau_2 = \gamma_a \tau_1 + \frac{1 - \gamma_a^2}{\gamma_a} \frac{\mathbf{r}_1 \cdot \mathbf{u}_1^a}{u_a^2}, \quad (45)$$

where  $u_a^2 = \mathbf{u}_1^a \cdot \mathbf{u}_1^a = \mathbf{u}_2^a \cdot \mathbf{u}_2^a$ ,  $\gamma_a = \gamma(u_a)$  and  $\lambda_{T1}^a$  maps the perpendicular component (with respect to  $\mathbf{u}_2^a$ ) of its  $K_2$ -argument onto a vector of  $K_1$  perpendicular to  $\mathbf{u}_1^a$ , preserving the length of that component. Equations (44) and (45) must be considered on an equal footing with the inverse equations obtained interchanging indices 1 and 2:

$$\mathbf{r}_2 = \lambda_{T2}^a(\mathbf{r}_1) + \mathbf{u}_2^a \tau_2 - \frac{1}{\gamma_a} \frac{\mathbf{r}_1 \cdot \mathbf{u}_1^a}{u_a^2} \mathbf{u}_2^a, \quad (46)$$

$$\tau_1 = \gamma_a \tau_2 + \frac{1 - \gamma_a^2}{\gamma_a} \frac{\mathbf{r}_2 \cdot \mathbf{u}_2^a}{u_a^2}. \quad (47)$$

The formulas for the other two transformations can be immediately deduced from the preceding ones by changing on them the indices  $a$ , for the transformation, and the related pair 1–2, for the linked spaces, to the ones desired.

This set of equations will permit us to find an universal form for the relation between the parameter  $\gamma$  and the relative velocity for any transformation as well as formulas relating the relative velocities of the three transformations. For that purposes it is easier to study the motion of the origin of each space from the other two spaces together with the transformation linking the last

ones. According to the equations defining the transformation  $T^c$ , the events experienced by the chosen origin of  $K_3$ ,  $\mathbf{r}_3 = 0$ , appear in  $K_1$  as

$$\mathbf{r}_1 = \mathbf{u}_1^c \tau_1, \quad (48)$$

$$\tau_1 = \gamma_c \tau_3. \quad (49)$$

In the same way,  $\mathbf{r}_3 = 0$  appears in  $K_2$  as

$$\mathbf{r}_2 = \mathbf{u}_2^b \tau_2, \quad (50)$$

$$\tau_2 = \gamma_b \tau_3, \quad (51)$$

obtained from the equations of the transformation  $T^b$ . The records of such events registered in  $K_1$  and  $K_2$  are related by the equations of the transformation  $T^a$  given above. By substitution of Eqs. (48)–(51) into Eq. (44) we get a relation among relative velocities:

$$\mathbf{u}_1^c = \frac{\gamma_b}{\gamma_c} \lambda_{T_1}^a(\mathbf{u}_2^b) + \left[ 1 - \frac{\gamma_b}{\gamma_a \gamma_c} \frac{\mathbf{u}_2^b \cdot \mathbf{u}_2^a}{\mathbf{u}_a^2} \right] \mathbf{u}_1^a. \quad (52)$$

Similarly, by eliminating times among Equations (45), (49) and (51), the former becomes

$$\frac{\gamma_b}{\gamma_c} = \gamma_a + \frac{1 - \gamma_a^2}{\gamma_a} \frac{\mathbf{u}_1^c \cdot \mathbf{u}_1^a}{\mathbf{u}_a^2}. \quad (53)$$

By studying in an identical way the motion of the origin of  $K_2$  from  $K_1$  and  $K_3$  and the equations of the transformation  $T^c$  we arrive to relations similar to the previous ones, namely

$$\mathbf{u}_1^a = \frac{\gamma_b}{\gamma_a} \lambda_{T_1}^c(\mathbf{u}_3^b) + \left[ 1 - \frac{\gamma_b}{\gamma_a \gamma_c} \frac{\mathbf{u}_3^b \cdot \mathbf{u}_3^c}{\mathbf{u}_c^2} \right] \mathbf{u}_1^c. \quad (54)$$

and

$$\frac{\gamma_b}{\gamma_a} = \gamma_c + \frac{1 - \gamma_c^2}{\gamma_c} \frac{\mathbf{u}_1^c \cdot \mathbf{u}_1^a}{\mathbf{u}_c^2}. \quad (55)$$

Results similar to those given by Eqs. (52)–(55) may be readily obtained from the consideration of the motions of the other possible pairs of origins.



Finally, by eliminating the product  $\mathbf{u}_1^c \cdot \mathbf{u}_1^a$  between Eqs. (53) and (55), and through analogous calculations on the other mentioned results, we arrive to the important identities

$$\frac{1 - \gamma_a^2}{\gamma_a^2} \frac{1}{u_a^2} = \frac{1 - \gamma_b^2}{\gamma_b^2} \frac{1}{u_b^2} = \frac{1 - \gamma_c^2}{\gamma_c^2} \frac{1}{u_c^2}. \quad (56)$$

Hence, the quantity  $(1 - \gamma^2)/(\gamma u)^2$  has a universal value irrespective the transformation it refers to. By denoting it as  $\alpha$ , we get the advertised expression for  $\gamma(u)$ :

$$\gamma = \frac{1}{\sqrt{1 + \alpha u^2}}. \quad (57)$$

The theory does not yield the actual value for  $\alpha$  that will have to be obtained by experimental means. The simpler choice is  $\alpha = 0$  leading to Galileo transformation, briefly considered in Section 3. The choice  $\alpha = -1/c^2$  corresponds to Lorentz transformation, studied in Section 4. The possibility of a positive value for  $\alpha$  can be discarded on causality arguments. See Lévy–Leblond (1976) for more details.

#### SUMMARY OF SPACE AND TIME TRANSFORMATIONS

For later use we collect here the results obtained in this Section. The mixed transformation formulas between  $K$  and  $K'$ , transcribed with an obvious change in notation from Eqs. (44) and (45), where the expression for  $\gamma(u)$  given in Eq. (57) is used, take the general form:

$$\mathbf{r}' = \lambda'_T(\mathbf{r}) + \left( \tau' - \frac{1}{\gamma} \frac{\mathbf{r} \cdot \mathbf{u}}{u^2} \right) \mathbf{u}', \quad (58)$$

$$\tau = \gamma \tau' + \alpha \gamma \mathbf{r}' \cdot \mathbf{u}'. \quad (59)$$

For the standard transformation, the corresponding formulas are

$$\mathbf{r}' = \lambda'_T(\mathbf{r}) - \gamma \frac{\mathbf{r} \cdot \mathbf{u}}{u^2} \mathbf{u}' + \gamma \mathbf{u}' \tau, \quad (60)$$

$$\tau' = \gamma \tau + \alpha \gamma \mathbf{r} \cdot \mathbf{u}. \quad (61)$$

The important formula for velocity addition is easily obtained from the (inverse of the) previous relations as:

$$\mathbf{v} = \frac{\frac{1}{\gamma}\lambda_T(\mathbf{v}') + \left(1 - \frac{\mathbf{v}' \cdot \mathbf{u}'}{u^2}\right) \mathbf{u}}{1 + \alpha \mathbf{v}' \cdot \mathbf{u}'}, \quad (62)$$

where  $\mathbf{v} = d\mathbf{r}/d\tau$  and  $\mathbf{v}' = d\mathbf{r}'/d\tau'$  are the measured velocities of a particle in  $K$  and  $K'$ , respectively.

These equations, together with the method used for obtaining them, exhibit certain formal aspects which are worth to be emphasized.

1. Only intrinsic objects belonging to each space are used.
2. The mathematical formalism keeps each object separate in its own space. Thus, in Equation (62) a vector of  $K$  is written as the sum of two vectors of the same space, and a numerical coefficient is obtained as the dot product of two vectors of the other space.

### 3 Galilean transformation. Absolute space

If one takes  $\alpha = 0$  then, irrespective of the value of the relative velocity,  $\gamma = 1$ . Hence, from Eq. (59), the formula for time transformation is, simply,

$$\tau = \tau', \quad (63)$$

showing that the time elapsed between two events is the same in any space of reference. That is the old concept of absolute time, implicit in Newton's Mechanics. The transformation formula for position is, from Eq. (58),

$$\mathbf{r}' = \lambda'_T(\mathbf{r}) - \left(\frac{\mathbf{r} \cdot \mathbf{u}}{u^2}\right) \mathbf{u}' + \mathbf{u}'\tau, \quad (64)$$

which, together with Equation (63), are the Galilean transformation formulas between the spaces  $K$  and  $K'$ . The classical law for velocity addition can now readily be obtained from Equation (62), with  $\gamma = 1$ , as

$$\mathbf{v} = \lambda_T(\mathbf{v}') + \left(1 - \frac{\mathbf{v}' \cdot \mathbf{u}'}{u^2}\right) \mathbf{u}. \quad (65)$$

As it is easily shown, the linear mapping  $\lambda$ , and not only  $\lambda_T$ , is an isometry, i. e.,  $\lambda(\mathbf{r}') \cdot \lambda(\mathbf{r}') = \mathbf{r}' \cdot \mathbf{r}'$ , and thus defines a metric isomorphism between the vectors in  $\mathcal{V}'$  and  $\mathcal{V}$ . Therefore, a metric isomorphism between  $K'$  and  $K$  can be established which makes possible to identify both spaces at any time according to the rules:

1. Place  $Q'$  on  $\Lambda_t(Q') = Q + (t - t_0)\mathbf{u}$ ;
2. Place  $\mathbf{u}'$  on  $\lambda(\mathbf{u}') = -\mathbf{u}$ ; and
3. Place a vector  $\mathbf{p}' \in T'$  on  $\lambda(\mathbf{p}')$ .

In this way, any point of  $K'$  is identified with its trace in  $K$ . The idea of an absolute space, as defined in Desloge (1982), comes from the possibility of this identification. We should point out that this identification is possible at any speed, even at relative rest. This is the reason why figures showing two frames of reference in relative motion seem so obvious: in spite of the inherent limitations of a still picture forcing us to draw not only  $K$  but also  $K'$  at rest, such identification is always allowed. On the other hand, since this identification is not possible in the spacetime of Minkowski when  $K'$  is moving, such simple figures cannot properly be drawn, situation that we discuss in the following Section.

We may fuse  $K$  and  $K'$  into one space by setting  $\mathbf{r}' \equiv \lambda(\mathbf{r}')$  and  $-\mathbf{u} \equiv \lambda(\mathbf{u}')$  and obtain the well known transformation formula

$$\mathbf{r} = \tau\mathbf{u} + \mathbf{r}'. \quad (66)$$

This classical result entails a trap very difficult to escape from. The Galilean transformation formula (66), an innocent vector addition, seems to be a direct consequence of the affine character of the (absolute) space. When it was shown that this was not an accurate physical law it resulted rather natural to doubt the affine property of the space by introducing an spatial anisotropy along the motion of the body of reference. That idea was formulated as the FitzGerald contraction hypothesis. If, on the contrary, we keep the separation between spaces and write the Galilean transformation as Equation (64), consequence of an absolute time, the jump to special relativity is easier since there is more room to look for a solution elsewhere. Einstein solved the difficulty by questioning the absolute character of time in his celebrated 1905 paper.

## 4 Lorentz transformation

By choosing a suitable negative value for  $\alpha$  one arrives to Lorentz transformation. Experimentally, one finds that, actually,  $\alpha = -1/c^2$ , where  $c$  is the speed of light in a vacuum. Accordingly, the actual dependence of  $\gamma$  on  $u$  is

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (67)$$

The formulas for the mixed Lorentz transformations are Eqs. (58) and (59) with this value for  $\gamma$ , i. e.,

$$\mathbf{r} = \lambda_T(\mathbf{r}') - \left( \frac{1}{\gamma} \frac{\mathbf{r}' \cdot \mathbf{u}'}{u^2} \right) \mathbf{u} + \mathbf{u} \tau, \quad (68)$$

$$\tau' = \gamma \left( \tau - \frac{\mathbf{r} \cdot \mathbf{u}}{c^2} \right). \quad (69)$$

The inverse transformation formulas are readily obtained by exchanging primed and unprimed symbols in the later expressions. From these equations we get the standard Lorentz transformation

$$\mathbf{r} = \lambda_T(\mathbf{r}') + \gamma \left( \tau' - \frac{\mathbf{r}' \cdot \mathbf{u}'}{u^2} \right) \mathbf{u}, \quad (70)$$

$$\tau = \gamma \left( \tau' - \frac{\mathbf{r}' \cdot \mathbf{u}'}{c^2} \right). \quad (71)$$

Now, from Equation (68), it can be seen that

$$\lambda(\mathbf{r}') \cdot \lambda(\mathbf{r}') = \mathbf{r}' \cdot \mathbf{r}' - \frac{(\mathbf{r}' \cdot \mathbf{u}')^2}{c^2}, \quad (72)$$

showing that  $\lambda$  is not an isometric mapping. This fact has the important consequence for the spaces  $K$  and  $K'$  that they cannot be identified to each other. Therefore, figures mixing two spaces, even if one-dimensional, are impossible, and only figures in spacetime can be drawn if one insists in showing both spaces on them.

## 5 Extending the method to two classical problems

The intrinsic method based on the concept of tracer mapping may be used in more general situations. As examples of that we have chosen two classical

problems, the analysis of which is, in our opinion, more transparent when the proposed method is used.

### 5.a Intrinsic definition of angular velocity

In Newtonian Mechanics, the spaces of two inertial bodies of reference,  $K$  and  $K'$ , may be metrically identified since the mapping  $\lambda$  preserves the magnitude of vectors. This is true independently of the relative velocity between the bodies. This fact makes possible to study the transformations between two spaces of reference having *arbitrary motions*. As before,  $\Lambda_t : K' \rightarrow K$  is the tracer mapping of  $K'$  on  $K$ .

**Postulate** *In Newtonian mechanics, the tracer mapping  $\Lambda_t$  between two spaces  $K$  and  $K'$ , having arbitrary relative motion, is affine and isometric.*

We take as origins the arbitrary points  $Q$  of  $K$  and  $Q'$  of  $K'$ . As before, we write the condition for the mapping  $\Lambda_t$  to be affine, at any time, as

$$\Lambda_t(P') = \Lambda_t\left(Q' + \overrightarrow{Q'P'}\right) = \Lambda_t(Q') + \lambda_t\left(\overrightarrow{Q'P'}\right), \quad (73)$$

but now we cannot take  $\lambda_t$  as time independent since different points of  $K'$  have different velocities. Equation (73) in vector form is

$$\mathbf{r} = \mathbf{R} + \lambda_t(\mathbf{r}') \quad (74)$$

where

$$\mathbf{r}(t) = \overrightarrow{Q\Lambda_t(P')}, \mathbf{R}(t) = \overrightarrow{Q\Lambda_t(Q')} \text{ and } \mathbf{r}'(t) = \overrightarrow{Q'P'}. \quad (75)$$

By differentiating (74) with respect to time, now absolute, we obtain a relation between velocities,

$$\mathbf{v} = \mathbf{V} + \dot{\lambda}_t(\mathbf{r}') + \lambda_t(\mathbf{v}'(t)), \quad (76)$$

where  $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ ,  $\mathbf{V}(t) = \dot{\mathbf{R}}(t)$  and  $\mathbf{v}'(t) = \dot{\mathbf{r}}'(t)$ . From now on, the  $t$  index in  $\lambda$  will be suppressed for a simpler notation.

Let us introduce the operator  $\Omega : \mathcal{V} \rightarrow \mathcal{V}$  defined as

$$\Omega = \dot{\lambda} \circ \lambda^{-1}. \quad (77)$$

By direct differentiation of the identities  $\lambda \circ \lambda^{-1} = \text{id}$  and  $\lambda^{-1}(\mathbf{x}) \cdot \lambda^{-1}(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ , we obtain the relation:

$$\Omega(\mathbf{x}) \cdot \mathbf{y} = -\mathbf{x} \cdot \Omega(\mathbf{y}),$$

showing that  $\Omega$  is an antisymmetric operator.

In terms of this operator, Equation (76) can be rewritten as

$$\mathbf{v} = \mathbf{V} + \Omega[\lambda(\mathbf{r}')] + \lambda(\mathbf{v}'). \quad (78)$$

Let  $\boldsymbol{\omega}$  be the vector associated to the antisymmetric operator  $\Omega$ , *i. e.*  $\boldsymbol{\omega} \times \mathbf{y} := \Omega(\mathbf{y})$  [see Crampin *et al.* (1986), p. 97]. Now, Equation (78) can be written in the more familiar form

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times [\lambda(\mathbf{r}')] + \lambda(\mathbf{v}'). \quad (79)$$

Hence,  $\boldsymbol{\omega}$  is the usual angular velocity and reasonably we can call  $\Omega$  the angular velocity operator.

As an important application of the angular velocity operator, we are going to derive an intrinsic expression for Coriolis theorem (see Dede *et al.* (1996) for an unconventional derivation of that theorem based on an interesting graphical trick, a kind of our tracer mapping). For that aim, we time differentiate Equation (78) and obtain

$$\begin{aligned} \mathbf{a} &= \dot{\mathbf{V}} + \lambda(\dot{\mathbf{v}}') + \dot{\Omega}[\lambda(\mathbf{r}')] + \Omega[\dot{\lambda}(\mathbf{r}')] \\ &\quad + \Omega[\lambda(\mathbf{v}')] + \dot{\lambda}(\mathbf{v}') \\ &= \mathbf{A} + \lambda(\mathbf{a}') + \dot{\Omega}[\lambda(\mathbf{r}')] + \Omega\left[\dot{\lambda} \circ \lambda^{-1} \circ \lambda(\mathbf{r}')\right] \\ &\quad + \Omega[\lambda(\mathbf{v}')] + \dot{\lambda} \circ \lambda^{-1} \circ \lambda(\mathbf{v}'). \end{aligned}$$

that, by using the definition of the angular velocity, can be written as

$$\mathbf{a} = \mathbf{A} + \lambda(\mathbf{a}') + \dot{\Omega}[\lambda(\mathbf{r}')] + \Omega\{\Omega[\lambda(\mathbf{r}')]\} + 2\Omega[\lambda(\mathbf{v}')], \quad (80)$$

the advertised Coriolis theorem. In this form, we see clearly which space each quantity belongs to. Besides, the derivation is straightforward and intrinsic. Finally, it is worth pointing out that whenever quantities in  $K'$  are to be obtained from those in  $K$  or vice versa it suffices to use  $\lambda$  and its time derivatives instead of  $\Omega$ . The intrinsic definition of the angular velocity makes possible to formulate the kinematics and dynamics of the rigid body in a completely intrinsic way.

## 5.b Geometry of the rotating disk in special relativity

It is well known the important heuristic role played by the rotating disk in the formulation of General Relativity [see Stachel (1989)]. Here we approach the study of the geometry of the rotating disk in the framework of the Special Relativity in a similar way as in Møller (1952), but trying to clarify the nature both of the problem statement and its solution by using the intrinsic method previously introduced.

The first problem is to give a precise and meaningful definition of a rotating disk. On the one hand, as in the rest of this paper,  $K$  is an inertial body of reference. As such, the corresponding space is euclidean and provided with a set of synchronized clocks. On the other hand,  $K'$  designates the rotating disk which is made of a continuum of particles.

As before, we define the trace of any particle of the disk  $K'$  on  $K$  at time  $t$  as the point in  $K$  coinciding with it at that time. We call  $\Lambda_t$  the tracer mapping of  $K'$  on  $K$ ,

$$\Lambda_t : K' \rightarrow K.$$

**Definition:** We say that  $K'$  is a rotating disk around a point  $O$  of  $K$  if the trace of any particle  $P'$  of  $K'$  describes a uniform circular motion around  $O$ , with an angular velocity  $\omega$ , the same for every particle.

We define a rotation operator in  $K$  around the point  $O$  by an angle  $\theta$  as  $e^{i\theta}$ ,

$$\mathbf{r} \mapsto e^{i\theta} \mathbf{r}. \quad (81)$$

Let  $P'$  be a fixed point in  $K'$ . By definition, the vector  $\overrightarrow{O\Lambda_t(P')}$  can be seen as the rotated  $\overrightarrow{O\Lambda_0(P')}$  by the angle  $\omega t$ , thus

$$\overrightarrow{O\Lambda_t(P')} = e^{i\omega t} \overrightarrow{O\Lambda_0(P')}. \quad (82)$$

The velocity of  $P'$  at  $t$  is

$$\mathbf{V}(P', t) = \frac{\partial}{\partial t} \overrightarrow{O\Lambda_t(P')} = i\omega e^{i\omega t} \overrightarrow{O\Lambda_0(P')} = i\omega \overrightarrow{O\Lambda_t(P')}, \quad (83)$$

where  $i$  rotates vectors by an angle of  $\pi/2$  in the positive sense. This result can be stated *à la Euler* as the velocity of the disk particle coinciding at  $t$  with the point  $P$  of  $K$  given as  $\overrightarrow{OP} = \overrightarrow{O\Lambda_t(P')}$ , i. e., the trace of  $P'$  at  $t$ . Thus

$$\mathbf{v}(P) = i\omega \overrightarrow{OP} = \mathbf{V}(P', t) \text{ provided } P = \Lambda_t(P'). \quad (84)$$

Now, we are going to construct the disk spatial metric. For that aim we introduce a collection of inertial bodies, comoving each with the velocity  $\mathbf{v}(P)$ . Thus, consider the reference body  $K'_P$  comoving with the disk particle passing by  $P$  at  $t$ . Being at relative rest, a little patch of  $K'$  around  $P'$  can be identified with a similar portion of  $K'_P$ , and thus the mapping  $\Lambda_t^P : K'_P \rightarrow K$  can be used for transforming the portion of  $K'$  into  $K$ . Therefore,

$$\Lambda_t^P(P' + d\mathbf{r}') = \Lambda_t^P(P') + \lambda^P(d\mathbf{r}').$$

Hence, the little displacement  $d\mathbf{r}$  at  $K$  and the corresponding  $d\mathbf{r}'$  at the disk are related as

$$d\mathbf{r} = \Lambda_t^P(P' + d\mathbf{r}') - \Lambda_t^P(P') = \lambda^P(d\mathbf{r}') \quad (85)$$

From Theorem 4,

$$\begin{aligned} d\mathbf{r}_T^2 &= d\mathbf{r}'_T^2 \\ d\mathbf{r}_U^2 &= \frac{1}{\gamma^2} d\mathbf{r}'_U^2, \end{aligned}$$

where, according to Equation (67),

$$\gamma = \left(1 - \frac{\omega^2}{c^2} r^2\right)^{-1/2}. \quad (86)$$

Hence,

$$d\mathbf{r}'^2 = d\mathbf{r}_T^2 + \gamma^2 d\mathbf{r}_U^2. \quad (87)$$

Thus, we have obtained the local metric of the disk in terms of the metric of its trace on  $K$ . This metric confers the disk a structure of a Riemannian manifold.

We parametrize the points of the disk manifold by introducing polar coordinates in  $K$ . The velocity of a point  $P(r, \theta)$  of the disk, according to Equation (84), is  $\mathbf{v}(P) = r\omega\hat{\mathbf{e}}_\theta$ . A small but arbitrary displacement from  $P$ ,  $d\mathbf{r} = dr\hat{\mathbf{e}}_r + r d\theta\hat{\mathbf{e}}_\theta$ , can be divided into parallel and perpendicular components respect to  $\mathbf{v}(P)$  as

$$d\mathbf{r}_U^2 = r^2 d\theta^2 \text{ and } d\mathbf{r}_T^2 = dr^2. \quad (88)$$



Then, the expression for the metric (87) in the chosen chart is

$$d\mathbf{r}'^2 = dr^2 + \gamma^2 r^2 d\theta^2, \quad (89)$$

where  $\gamma$ , as given by (86), depends on the radial coordinate  $r$ . Finally, the element of arc length is

$$dl' = \sqrt{d\mathbf{r}'^2} = \sqrt{dr^2 + \gamma^2 r^2 d\theta^2}. \quad (90)$$

Now, it is immediate to obtain the disk radius length from the element of arc by taking  $d\theta = 0$ :

$$R' = \int_0^R \sqrt{dr^2 + \gamma^2 r^2 0^2} = R$$

The disk rim length is obtained by integration of the arc element along  $r = R$ :

$$l' = \int_0^{2\pi} \sqrt{0^2 + \gamma^2 R^2 d\theta^2} = \frac{2\pi R}{\sqrt{1 - \frac{\omega^2}{c^2} R^2}}.$$

## 6 Conclusions

We have developed an intrinsic method for deriving space and time transformations. For that purpose, following a suggestion by Einstein, we associated a different space of reference to every reference body. In order to connect the space and time measurements in the spaces we introduced the tracer mapping, a simple concept fitted with a direct physical meaning, which makes possible to express with a greater clarity several ideas of the special relativity.

The intrinsic method shows some advantages we are going to summarize:

1. The space and time transformations were derived using no coordinate systems. Therefore, a clear distinction is now possible between a coordinate system and a body of reference.
2. The tracer mapping allows a sharp definition for the geometry of a moving body: It is nothing but the geometry of its trace. A reference space not equipped with clocks is an incomplete system to register events. Nevertheless, as seen in the discussion of the rotating disk, we were able to obtain its geometry by using the tracer mapping as it does not require any time measurement in that space.

3. The intrinsic method does not depend in any way on a notion of parallelism between both spaces of reference. Rather, the trace allows us to define that parallelism in an operational way. Note that, in the non-intrinsic derivations of the spacetime transformations, the coordinate axes of both reference frames are assumed parallel to each other. But, here we have not argued, for example, that  $\mathbf{u}$  and  $\mathbf{u}'$  are parallel because they belong to different spaces. We affirm, instead, that  $\mathbf{u}$  and  $\lambda(\mathbf{u}')$  are parallel.
4. Even though in the Galilean space and time one might identify the different spaces of reference, the lesson our derivation of Coriolis Theorem teaches us is that the tracer mapping between separated spaces is useful even in these cases, in which one could identify them.

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## References

- V. Berzi and V. Gorini, “Reciprocity Principle and the Lorentz Transformations,” *J. Math. Phys.* **10**, pp. 1518–1524 (1968).
- M. Crampin and F. A. E. Pirani, *Applicable Differential Geometry*, (Cambridge University Press, Cambridge, U. K. 1986).
- J. T. Cushing, “Vector Lorentz Transformations,” *American Journal of Physics* **35**, pp. 858–862 (1967).
- K. M. Dede, A. Demény and J. Kuti Darai, “A Coordinate Independent Approach to the Kinematics of Rotating Frames,” *American Journal of Physics*, **64**, pp. 482–484 (1996).
- M. Desloge, *Classical Mechanics*, Vol. 1 (John Wiley and Sons, New York, 1982), p. 4.

- A. Einstein, *The Meaning of Relativity* (1922; Princeton University Press, Princeton, N. J., 1953), p. 3.
- C. F. Frahm, “Representing Arbitrary Boosts for Undergraduates,” *American Journal of Physics* **47**, pp. 870–872 (1979).
- G. Galilei, *Dialogue Concerning the Two Chief World Systems–Ptolemaic and Copernican* (1632) English translation by S. Drake from the Italian original edition, (University of California, Berkeley, 1967), Second Dialogue.
- Marsden, J. E. and T. J. R. Hughes, *Mathematical Foundations of Elasticity*, (Dover, New York, 1994).
- J.–M. Lévy–Leblond, “One more Derivation of the Lorentz Transformation,” *American Journal of Physics*, **41**, pp. 271–277 (1976).
- T. Matolcsi, *Spacetime without reference frames*, (Akadémiai Kiadó, Budapest, 1993); T. Matolcsi and A. Gohér, *Spacetime without Reference Frames and its Application to the Thomas Rotation*, preprint (<http://www.cs.elte.hu/applanal/preprints/rotation.dvi>).
- C. Møller, *The Theory of Relativity*, (Oxford University Press, London, U. K. 1952), §84.
- S. Nishikawa, “Lorentz Transformation without the Direct Use of Einstein’s Postulates,” *Nuovo Cimento B* **112**, pp. 1175–1187 (1997).
- W. Rindler, *Essential relativity*, 2nd Ed. (Springer, New York, 1977) p. 40.
- H. M. Schwartz, “Deduction of the General Lorentz Transformations from a Set of Necessary Assumptions,” *American Journal of Physics* **52**, pp. 346–350 (1984).
- J. Stachel, “The Rigidly Rotating Disk as the ‘Missing Link’ in the History of General Relativity,” in *Einstein and the History of General Relativity*, D. Howard and J. Stachel, eds. (Birkhäuser, Boston, 1989).
- J. L. Synge, *Relativity, the Special Theory*, 2nd Ed. (North–Holland, Amsterdam, 1972), pp. 119–120.